

TURÁN TYPE INEQUALITIES FOR REGULAR COULOMB WAVE FUNCTIONS

ÁRPÁD BARICZ

ABSTRACT. Turán, Mitrinović-Adamović and Wilker type inequalities are deduced for regular Coulomb wave functions. The proofs are based on a Mittag-Leffler expansion for the regular Coulomb wave function, which may be of independent interest. Moreover, some complete monotonicity results concerning the Coulomb zeta functions and some interlacing properties of the zeros of Coulomb wave functions are given.

1. INTRODUCTION

The Coulomb wave function, which bears the name of the famous French physicist Charles Augustin de Coulomb (best known for his law describing the electrostatic interaction between electrically charged particles), is a solution of the Coulomb wave equation (or radial Schrödinger equation in the Coulomb potential) and it is used to describe the behavior of charged particles in a Coulomb potential. There is an extensive literature concerning the computation of the Coulomb wave function values, however, the zeros and other analytical properties have not been studied in detail. For more details we refer to the papers [Ik, MKCI] and to the references therein. We mention that recently, an important study on the Coulomb wave function was made by Štampach and Šťovíček [ŠŠ]. In this paper we present some new results on the Coulomb wave function, which may be useful for people working in special functions and mathematical physics. Our present paper belongs to the rich literature about Turán type inequalities on orthogonal polynomials and special functions, named after the Hungarian mathematician Paul Turán, and can be interpreted as the generalization of some of the results on Bessel functions of the first kind, obtained by Szász [Sz1, Sz2]. The paper is organized as follows: the next section is divided into four subsections and contains some Turán, Mitrinović-Adamović and Wilker type inequalities for the regular Coulomb wave function. The key tool in the proofs is a Mittag-Leffler expansion for the regular Coulomb wave function, which may be of independent interest. We also deduce some complete monotonicity results for the Coulomb zeta functions, which are defined by using the real zeros of the Coulomb wave functions. By using the Hadamard factorization of the Coulomb wave functions we also present some interlacing properties of the zeros of the Coulomb wave functions.

2. PROPERTIES OF THE REGULAR COULOMB WAVE FUNCTIONS

In this section our aim is to present the main results of this paper about the regular Coulomb wave function together with their proofs. The section is divided into four subsections.

2.1. Turán type inequalities for regular Coulomb wave functions. In order to obtain the main results of this subsection we use a Mittag-Leffler expansion for the regular Coulomb wave function together with the recurrence relations, and a result of Ross [Ro]. As we can see below the second main result of this subsection is a natural extension of a well-known result of Szász [Sz1, Sz2] for Bessel functions of the first kind. The next result, which may be of independent interest, is an immediate consequence of a result of Wimp [Wi] concerning confluent hypergeometric functions and it was recently rediscovered by Štampach and Šťovíček [ŠŠ], by using a different method. In both papers [ŠŠ, Wi] a new class of orthogonal polynomials associated with regular Coulomb wave functions is introduced. These polynomials play a

File: Coulomb.tex, printed: 2015-04-27, 0.25

2010 *Mathematics Subject Classification.* Primary 33C15, Secondary 26D07, 39C05.

Key words and phrases. Coulomb wave functions; Mittag-Leffler expansion; Turán, Mitrinović-Adamović, Wilker type inequality; Coulomb zeta functions; complete monotonicity; interlacing property of zeros.

★The research of Á. Baricz was supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences. The author is very grateful to Mourad E.H. Ismail for suggesting the study on the regular Coulomb wave function and also for his kind hospitality during the author's visit at Department of Mathematics of the City University of Hong Kong in September 2011. The author is also grateful to his colleague Ágoston Róth for the discussions about the growth order of the regular Coulomb wave function.

role analogous to that the Lommel polynomials have in the theory of Bessel functions of the first kind. However, it is worth to mention that Wimp's approach [Wi] is based on inversion of Stieltjes transforms, while Štampach and Štoviček [ŠŠ] used the eigenvalues of some Jacobi matrices.

Lemma 1. *Let $\rho, \eta \in \mathbb{R}$ and let $L > -3/2$, $L \neq -1$ if $\eta \neq 0$ and $L > -3/2$ if $\eta = 0$. Then the next Mittag-Leffler expansion is valid*

$$(2.1) \quad \frac{F_{L+1}(\eta, \rho)}{F_L(\eta, \rho)} = \frac{L+1}{\sqrt{(L+1)^2 + \eta^2}} \sum_{n \geq 1} \left[\frac{\rho}{x_{L,\eta,n}(x_{L,\eta,n} - \rho)} + \frac{\rho}{y_{L,\eta,n}(y_{L,\eta,n} - \rho)} \right],$$

where $x_{L,\eta,n}$ and $y_{L,\eta,n}$ are the n th positive and negative zeros of the Coulomb wave function $F_L(\eta, \rho)$.

Proof. Let ${}_1F_1$ denotes the Kummer confluent hypergeometric function. It is known that

$$F_L(\eta, \rho) = C_L(\eta) \rho^{L+1} e^{-i\rho} {}_1F_1(L+1 - i\eta, 2L+2; 2i\rho),$$

where

$$C_L(\eta) = \frac{2^L e^{-\frac{\pi\eta}{2}} |\Gamma(L+1 + i\eta)|}{\Gamma(2L+2)}.$$

By using the next result of Wimp [Wi, p. 892] for $c = 2L+2$, $\kappa = \eta$ and $z = 1/\rho$

$$\frac{{}_1F_1\left(\frac{c}{2} + 1 - i\kappa, c+2; \frac{2i}{z}\right)}{{}_1F_1\left(\frac{c}{2} - i\kappa, c; \frac{2i}{z}\right)} = \frac{c^2(c+1)}{c^2 + 4\kappa^2} \sum_{k \in \mathbb{Z} \setminus \{0\}} z_k^{-2} \frac{z}{z - z_k^{-1}},$$

where $\kappa, z \in \mathbb{R}$, $c > -1$ and $z_k, k \in \mathbb{Z} \setminus \{0\}$, are the zeros of the function ${}_1F_1(c/2 - i\kappa, c; 2iz)$, it follows that

$$\frac{F_{L+1}(\eta, \rho)}{F_L(\eta, \rho)} = \frac{C_{L+1}(\eta)}{C_L(\eta)} \frac{(L+1)^2(2L+3)}{(L+1)^2 + \eta^2} \sum_{n \geq 1} \left[\frac{\rho}{x_{L,\eta,n}(x_{L,\eta,n} - \rho)} + \frac{\rho}{y_{L,\eta,n}(y_{L,\eta,n} - \rho)} \right],$$

which by means of the relation [AS, p. 538] $L(2L+1)C_L(\eta) = \sqrt{L^2 + \eta^2} C_{L-1}(\eta)$ yields (2.1). We note that in the above formula of Wimp [Wi, p. 892] instead of the correct expression $K = c^2(c+1)/(c^2 + 4\kappa^2)$ it was used $K = c^2(c+1)/(c^2/4 + \kappa^2)$, and instead of the correct argument $2i/z$ it was i/z . This can be verified by using the fact that when $\eta = 0$ the Coulomb wave function reduces to Bessel function of the first kind, and by using the Mittag-Leffler expansion for Bessel functions of the first kind and the first Rayleigh sum of zeros of Bessel functions we would have contradiction.

Another way to obtain (2.1) is to consider the Hadamard infinite product expansion [ŠŠ, eq. 76]

$$(2.2) \quad F_L(\eta, \rho) = C_L(\eta) \rho^{L+1} e^{\frac{\eta\rho}{L+1}} \prod_{n \geq 1} \left(1 - \frac{\rho}{\rho_{L,\eta,n}} \right) e^{\frac{\rho}{\rho_{L,\eta,n}}},$$

where $\rho_{L,\eta,n}$ is the n th zero of the Coulomb wave function. Logarithmic derivation yields

$$\frac{F'_L(\eta, \rho)}{F_L(\eta, \rho)} = \frac{L+1}{\rho} + \frac{\eta}{L+1} - \sum_{n \geq 1} \frac{\rho}{\rho_{L,\eta,n}(\rho_{L,\eta,n} - \rho)},$$

which in view of the recurrence relation [AS, p. 539]

$$(2.3) \quad (L+1)F'_L(\eta, \rho) = \left[\frac{(L+1)^2}{\rho} + \eta \right] F_L(\eta, \rho) - \sqrt{(L+1)^2 + \eta^2} F_{L+1}(\eta, \rho)$$

yields

$$\frac{F_{L+1}(\eta, \rho)}{F_L(\eta, \rho)} = \frac{L+1}{\sqrt{(L+1)^2 + \eta^2}} \sum_{n \geq 1} \frac{\rho}{\rho_{L,\eta,n}(\rho_{L,\eta,n} - \rho)}.$$

Now, taking into account that the zeros $\rho_{L,\eta,n}$ can be separated into positive and negative zeros, the proof of (2.1) is done. \square

It is worth to mention that if $\eta = 0$, then (2.1) reduces to the next well-known Mittag-Leffler expansion

$$\frac{F_{L+1}(0, \rho)}{F_L(0, \rho)} = \frac{J_{L+3/2}(\rho)}{J_{L+1/2}(\rho)} = \sum_{n \geq 1} \frac{2\rho}{j_{L+1/2,n}^2 - \rho^2},$$

where $L > -3/2$, J_L stands for the Bessel function of the first kind of order L and $j_{L,n}$ is the n th positive zero of the Bessel function J_L . Here we used that for each natural n we have $x_{L,0,n} = -y_{L,0,n} = j_{L+1/2,n}$, that is, the corresponding negative and positive zeros of the Bessel function of the first kind are symmetric with respect to the origin.

Now, we are ready to present the first set of results concerning the Turán type inequalities for the regular Coulomb wave function. Three kind of Turánians are considered and the results are mainly based on the Mittag-Leffler expansion (2.1). Our first main result is the following theorem.

Theorem 1. *The following assertions are true:*

- a.** *If $L, \eta > 0$, $0 < \rho < L(L+1)/\eta$, $\rho < x_{L,\eta,1}$ or $-3/2 < L < -1$, $\eta > 0$, $0 < \rho < L(L+1)/\eta$, $\rho < x_{L,\eta,1}$ or $\eta \leq 0$, $L \geq 0$ and $0 < \rho < x_{L,\eta,1}$, then*

$$(2.4) \quad F_L^2(\eta, \rho) - F_{L-1}(\eta, \rho)F_{L+1}(\eta, \rho) \geq 0.$$

- b.** *If $L, \eta > 0$, $L(L+1)/\eta \leq \rho < x_{L-1,\eta,1}$ or $-3/2 < L < -1$, $\eta > 0$, $L(L+1)/\eta \leq \rho < x_{L-1,\eta,1}$ or $-1 < L < 0$, $\eta < 0$, $L(L+1)/\eta \leq \rho < x_{L-1,\eta,1}$ then*

$$\frac{\sqrt{L^2 + \eta^2}}{L} F_L^2(\eta, \rho) - \frac{\sqrt{(L+1)^2 + \eta^2}}{L+1} F_{L-1}(\eta, \rho)F_{L+1}(\eta, \rho) \geq 0.$$

- c.** *If $L > -1$, $\eta \in \mathbb{R}$, $\rho^2 \leq (L^3 + 1)/(L^2 + \eta^2)$, $\eta/(L(L+1)) - 1/\rho > 0$ and $0 < \rho < x_{L-1,\eta,1}$, then*

$$F_L^2(\eta, \rho) - \frac{\sqrt{L^2 + \eta^2} \sqrt{(L+1)^2 + \eta^2}}{L(L+1)} F_{L-1}(\eta, \rho)F_{L+1}(\eta, \rho) \geq 0.$$

Proof. **a.** By using the recurrence relation [AS, p. 539]

$$(2.5) \quad LF_L'(\eta, \rho) = \sqrt{L^2 + \eta^2} F_{L-1}(\eta, \rho) - \left(\frac{L^2}{\rho} + \eta \right) F_L(\eta, \rho)$$

and (2.3), we obtain

$$\frac{{}_1\Delta_{L,\eta}(\rho)}{F_L^2(\eta, \rho)} = a_{L,\eta}(\rho) - b_{L,\eta}(\rho) \frac{F_L'(\eta, \rho)}{F_L(\eta, \rho)} + c_{L,\eta} \left[\frac{F_L'(\eta, \rho)}{F_L(\eta, \rho)} \right]^2,$$

where

$$a_{L,\eta}(\rho) = 1 - \frac{\left(\frac{L^2}{\rho} + \eta \right) \left[\frac{(L+1)^2}{\rho} + \eta \right]}{\sqrt{L^2 + \eta^2} \sqrt{(L+1)^2 + \eta^2}}, \quad b_{L,\eta}(\rho) = \frac{\frac{L(L+1)}{\rho} - \eta}{\sqrt{L^2 + \eta^2} \sqrt{(L+1)^2 + \eta^2}},$$

$$c_{L,\eta} = \frac{L(L+1)}{\sqrt{L^2 + \eta^2} \sqrt{(L+1)^2 + \eta^2}}$$

and ${}_1\Delta_{L,\eta}(\rho)$ stands for the Turán expression, defined by

$${}_1\Delta_{L,\eta}(\rho) = F_L^2(\eta, \rho) - F_{L-1}(\eta, \rho)F_{L+1}(\eta, \rho).$$

Now, taking into account that the Coulomb wave function is a particular solution of the Coulomb differential equation [AS, p. 538]

$$(2.6) \quad w''(\rho) + \left[1 - \frac{2\eta}{\rho} - \frac{L(L+1)}{\rho^2} \right] w(\rho) = 0,$$

we get

$$(2.7) \quad \left[\frac{F_L'(\eta, \rho)}{F_L(\eta, \rho)} \right]^2 = \frac{L(L+1)}{\rho^2} + \frac{2\eta}{\rho} - 1 - \left[\frac{F_L'(\eta, \rho)}{F_L(\eta, \rho)} \right]',$$

which in turn implies that

$$\frac{{}_1\Delta_{L,\eta}(\rho)}{F_L^2(\eta, \rho)} = d_{L,\eta}(\rho) - b_{L,\eta}(\rho) \frac{F_L'(\eta, \rho)}{F_L(\eta, \rho)} - c_{L,\eta} \left[\frac{F_L'(\eta, \rho)}{F_L(\eta, \rho)} \right]',$$

where

$$d_{L,\eta}(\rho) = 1 - \frac{L(L+1) + \frac{\eta}{\rho} + \eta^2}{\sqrt{L^2 + \eta^2} \sqrt{(L+1)^2 + \eta^2}}.$$

Moreover, by using the recurrence relation (2.3) and the Mittag-Leffler expansion (2.1), it follows

$$\frac{F_L'(\eta, \rho)}{F_L(\eta, \rho)} = \frac{L+1}{\rho} + \frac{\eta}{L+1} - \sum_{n \geq 1} \left[\frac{\rho}{x_{L,\eta,n}(x_{L,\eta,n} - \rho)} + \frac{\rho}{y_{L,\eta,n}(y_{L,\eta,n} - \rho)} \right]$$

and

$$(2.8) \quad \left[\frac{F_L'(\eta, \rho)}{F_L(\eta, \rho)} \right]' = -\frac{L+1}{\rho^2} - \sum_{n \geq 1} \left[\frac{1}{(x_{L,\eta,n} - \rho)^2} + \frac{1}{(y_{L,\eta,n} - \rho)^2} \right].$$

Consequently we have

$$\begin{aligned} \frac{{}_1\Delta_{L,\eta}(\rho)}{F_L^2(\eta, \rho)} &= e_{L,\eta} + b_{L,\eta}(\rho) \sum_{n \geq 1} \left[\frac{\rho}{x_{L,\eta,n}(x_{L,\eta,n} - \rho)} + \frac{\rho}{y_{L,\eta,n}(y_{L,\eta,n} - \rho)} \right] \\ &\quad + c_{L,\eta} \sum_{n \geq 1} \left[\frac{1}{(x_{L,\eta,n} - \rho)^2} + \frac{1}{(y_{L,\eta,n} - \rho)^2} \right], \end{aligned}$$

where

$$e_{L,\eta} = 1 - \frac{L\sqrt{(L+1)^2 + \eta^2}}{(L+1)\sqrt{L^2 + \eta^2}}.$$

Note that for all $L \geq 0$ or $-3/2 < L < -1$ and $\eta \in \mathbb{R}$ we have $c_{L,\eta} \geq 0$ and $e_{L,\eta} \geq 0$. Thus ${}_1\Delta_{L,\eta}(\rho)$ is positive if $L, \eta > 0$, $0 < \rho < L(L+1)/\eta$, $\rho < x_{L,\eta,1}$ or if $-3/2 < L < -1$, $\eta > 0$, $0 < \rho < L(L+1)/\eta$, $\rho < x_{L,\eta,1}$ or if $\eta \leq 0$, $L \geq 0$ and $0 < \rho < x_{L,\eta,1}$.

b. By using the recurrence relations (2.5) and (2.3) we obtain

$$F'_{L+1}(\eta, \rho)F_L(\eta, \rho) - F'_L(\eta, \rho)F_{L+1}(\eta, \rho) = {}_2\Delta_{L+1,\eta}(\rho) - \left[\frac{\eta}{(L+1)(L+2)} - \frac{1}{\rho} \right] F_L(\eta, \rho)F_{L+1}(\eta, \rho),$$

where

$${}_2\Delta_{L,\eta}(\rho) = \frac{\sqrt{L^2 + \eta^2}}{L} F_L^2(\eta, \rho) - \frac{\sqrt{(L+1)^2 + \eta^2}}{L+1} F_{L-1}(\eta, \rho)F_{L+1}(\eta, \rho).$$

On the other hand, according to [MKCI, Lemma 2.4] we have

$$\rho^2 \frac{\sqrt{(L+1)^2 + \eta^2}}{L+1} [F'_{L+1}(\eta, \rho)F_L(\eta, \rho) - F'_L(\eta, \rho)F_{L+1}(\eta, \rho)] = \sum_{n \geq 1} (2L + 2n + 1) F_{L+n}^2(\eta, \rho).$$

From this we obtain that

$$\frac{{}_2\Delta_{L,\eta}(\rho)}{F_{L-1}^2(\eta, \rho)} \geq \left[\frac{\eta}{L(L+1)} - \frac{1}{\rho} \right] \frac{F_L(\eta, \rho)}{F_{L-1}(\eta, \rho)}$$

and by using the Mittag-Leffler expansion (2.1), the right-hand side of the above inequality is positive if $L, \eta > 0$ and $L(L+1)/\eta \leq \rho < x_{L-1,\eta,1}$ or if $-3/2 < L < -1$, $\eta > 0$ and $L(L+1)/\eta \leq \rho < x_{L-1,\eta,1}$ or if $-1 < L < 0$, $\eta < 0$, $L(L+1)/\eta \leq \rho < x_{L-1,\eta,1}$.

c. Observe that (2.8) implies that for all $\eta, \rho \in \mathbb{R}$, $\rho \neq 0$ and $L \geq -1$ we have

$$D_{L,\eta}(\rho) = F''_L(\eta, \rho)F_L(\eta, \rho) - F'_L(\eta, \rho)F'_L(\eta, \rho) \leq 0.$$

Now, by using the recurrence relations (2.5) and (2.3) and also the fact that $F_L(\eta, \rho)$ satisfies the Coulomb differential equation (2.6), we obtain

$$D_{L,\eta}(\rho) = f_{L,\eta}(\rho)F_L^2(\eta, \rho) + \frac{1}{c_{L,\eta}}F_{L-1}(\eta, \rho)F_{L+1}(\eta, \rho) + \left[\frac{\eta}{L(L+1)} - \frac{1}{\rho} \right] F_{L-1}(\eta, \rho)F_L(\eta, \rho),$$

where

$$f_{L,\eta}(\rho) = \frac{L}{\rho^2} - 1 - \frac{\eta^2}{L^2}.$$

If $L > -1$, $\eta \in \mathbb{R}$ and $(L^3 + 1)/(L^2 + \eta^2) \geq \rho^2$, then we have that $f_{L,\eta}(\rho) \geq -1$ and consequently we have

$$0 \geq D_{L,\eta}(\rho) \geq -{}_3\Delta_{L,\eta}(\rho) + \left[\frac{\eta}{L(L+1)} - \frac{1}{\rho} \right] F_{L-1}(\eta, \rho)F_L(\eta, \rho),$$

where

$${}_3\Delta_{L,\eta}(\rho) = F_L^2(\eta, \rho) - \frac{\sqrt{L^2 + \eta^2}\sqrt{(L+1)^2 + \eta^2}}{L(L+1)} F_{L-1}(\eta, \rho)F_{L+1}(\eta, \rho).$$

But the above inequality is equivalent to

$$\frac{{}_3\Delta_{L,\eta}(\rho)}{F_{L-1}^2(\eta, \rho)} \geq \left[\frac{\eta}{L(L+1)} - \frac{1}{\rho} \right] \frac{F_L(\eta, \rho)}{F_{L-1}(\eta, \rho)}$$

and by using again the Mittag-Leffler expansion (2.1), the right-hand side of the above inequality is positive if $\eta/(L(L+1)) - 1/\rho > 0$ and $0 < \rho < x_{L-1,\eta,1}$. With this the proof is complete. \square

Now, let us consider the notations

$$B_{L,\eta}(\rho) = \frac{L\sqrt{(L+1)^2 + \eta^2}}{(2L+1)\left[\frac{L(L+1)}{\rho} + \eta\right]} \quad \text{and} \quad C_{L,\eta}(\rho) = \frac{(L+1)\sqrt{L^2 + \eta^2}}{(2L+1)\left[\frac{L(L+1)}{\rho} + \eta\right]}.$$

In what follows we show that if $L \geq 0$, $\eta \leq 0$, then the restriction $\rho < x_{L,\eta,1}$ in the Turán type inequality (2.4) can be removed. Moreover, we show that in this case the inequality (2.4) can be improved.

Theorem 2. *If $n \in \{0, 1, \dots\}$ and $L \geq -3/2$, $L \neq -1$, $\rho > 0$, $\eta \in \mathbb{R}$, $\eta \neq 0$ or $L > -3/2$, $\rho > 0$ and $\eta = 0$, then*

$$(2.9) \quad F_{L+n}^2(\eta, \rho) - F_{L+n-1}(\eta, \rho)F_{L+n+1}(\eta, \rho) = -\frac{\Theta C_{L+n,\eta}(\rho)}{C_{L+n,\eta}(\rho)} F_{L+n}^2(\eta, \rho) - \sum_{i=1}^{\infty} \frac{B_{L+n+1,\eta}(\rho)B_{L+n+2,\eta}(\rho) \cdots B_{L+n+i+1,\eta}(\rho)}{C_{L+n,\eta}(\rho)C_{L+n+1,\eta}(\rho) \cdots C_{L+n+i,\eta}(\rho)} \Theta(B_{L+n+i-1,\eta}(\rho)C_{L+n+i,\eta}(\rho)) F_{L+n+i}^2(\eta, \rho),$$

where Θ is the forward difference operator defined by $\Theta A_n = A_{n+1} - A_n$.

In particular, for all $L \geq 0$, $\eta \leq 0$ and $\rho > 0$ the following sharp Turán type inequality is valid

$$(2.10) \quad F_L^2(\eta, \rho) - F_{L-1}(\eta, \rho)F_{L+1}(\eta, \rho) \geq \left[1 - \frac{L(2L+1)\sqrt{(L+1)^2 + \eta^2}}{(L+1)(2L+3)\sqrt{L^2 + \eta^2}}\right] F_L^2(\eta, \rho).$$

It is important to mention here that when $\eta = 0$ the Turán type inequalities (2.4) and (2.10) reduce to known results of Szász [Sz1]. More precisely, since [AS, p. 542] $F_L(0, \rho) = \sqrt{\frac{\pi\rho}{2}} J_{L+1/2}(\rho)$, the Turán type inequalities (2.4) and (2.10) for $\eta = 0$ and $L+1/2 = \nu$ become

$$J_\nu^2(\rho) - J_{\nu-1}(\rho)J_{\nu+1}(\rho) \geq 0,$$

$$J_\nu^2(\rho) - J_{\nu-1}(\rho)J_{\nu+1}(\rho) \geq \frac{1}{\nu+1} J_\nu^2(\rho),$$

where $\nu \geq 1/2$ and $\rho > 0$. For more details on Turán type inequalities for Bessel functions and other generalizations we refer to the papers [BP, BI, JB, KS, La, Pa, Sk, Sz2, TN] and the references therein.

The proof of the above theorem is based on the next result of Ross [Ro, Theorem 3].

Lemma 2. *Let I be an interval and let $\{y_n\}_{n \geq 0}$ be a sequence of functions of real variable x , which is uniformly bounded in n for each $x \in I$. If these functions satisfy*

$$y_n(x) = B_n y_{n+1}(x) + C_n y_{n-1}(x),$$

where B_n and C_n are functions of x , $x \in I$, with the property that $C_n(x) \neq 0$, $B_n(x) \rightarrow 0$ and $\prod_{i=1}^n |B_i(x)/C_i(x)|$ converges as $n \rightarrow \infty$ for all $x \in I$, then

$$(2.11) \quad y_n^2(x) - y_{n-1}(x)y_{n+1}(x) = -\frac{\Theta C_n}{C_n} y_n^2(x) - \sum_{i=1}^{\infty} \frac{B_{n+1}B_{n+2} \cdots B_{n+i+1}}{C_n C_{n+1} \cdots C_{n+i}} \Theta(B_{n+i-1}C_{n+i}) y_{n+i}^2(x).$$

For reader's convenience we note here that in formula (i) of [Ro, p. 28] the expression $B_n y_n$ should be written as $B_{n+1} y_n$, and in the main formula of [Ro, Theorem 3] the expression B_{n+i-1} should be written as B_{n+i+1} , just like in (2.11).

Proof of Theorem 2. In order to deduce the infinite sum representation of the Turánian of the Coulomb wave functions in Theorem 2 we shall use Lemma 2. According to the recurrence relation [AS, p. 539]

$$B_{L,\eta}(\rho)F_{L+1}(\eta, \rho) = F_L(\eta, \rho) - C_{L,\eta}(\rho)F_{L-1}(\eta, \rho)$$

we have

$$F_{L+n}(\eta, \rho) = B_{L+n,\eta}(\rho)F_{L+n+1}(\eta, \rho) + C_{L+n,\eta}(\rho)F_{L+n-1}(\eta, \rho).$$

Observe that when $n \in \{0, 1, \dots\}$ for $L > -3/2$, $L \neq -1$, $\rho \in \mathbb{R}$ and $\eta \in \mathbb{R}$, $\eta \neq 0$, or $L > -3/2$, $\rho \in \mathbb{R}$ and $\eta = 0$ we have $C_{L+n,\eta}(\rho) \neq 0$ and $B_{L+n,\eta}(\rho) \rightarrow 0$ as $n \rightarrow \infty$. Moreover, the product

$$\prod_{i=1}^n \frac{B_{L+i,\eta}(\rho)}{C_{L+i,\eta}(\rho)} = \prod_{i=1}^n \frac{(L+i)\sqrt{(L+i+1)^2 + \eta^2}}{(L+i+1)\sqrt{(L+i)^2 + \eta^2}} = \sqrt{\frac{1 + \frac{\eta^2}{(L+n+1)^2}}{1 + \frac{\eta^2}{(L+1)^2}}}$$

converges as $n \rightarrow \infty$ for all $L > -3/2$, $L \neq -1$, $\rho \in \mathbb{R}$ and $\eta \in \mathbb{R}$. We just need to check the uniform boundedness of the Coulomb wave function with respect to $L+n$. For this we use the asymptotic relation

$F_L(\eta, \rho) \sim C_L(\eta)\rho^{L+1}$ as $L \rightarrow \infty$. Note that according to [AS, p. 538] and [Ni, p. 43] for L positive integer we have

$$C_L(\eta) = \frac{2^L e^{-\frac{\pi\eta}{2}} |\Gamma(L+1+i\eta)|}{\Gamma(2L+2)} = \begin{cases} \frac{2^L}{(2L+1)!} \sqrt{\frac{2\pi \prod_{k=0}^L (k^2+\eta^2)}{\eta(e^{2\pi\eta}-1)}}, & \text{if } \eta \neq 0 \\ \frac{2^L L!}{(2L+1)!}, & \text{if } \eta = 0 \end{cases}.$$

Thus, by using the infinite product representation of the hyperbolic sine function [AS, p. 85] we get that for fixed $\eta \in \mathbb{R}$ and $\rho > 0$

$$C_L(\eta)\rho^{L+1} \rightarrow C_L(0)\rho^{L+1} \sqrt{\frac{2\sinh(\pi\eta)}{e^{2\pi\eta}-1}} = \frac{\sqrt{\pi}\rho^{L+1}}{2^{L-1}\Gamma(L+\frac{3}{2})} e^{-\frac{\pi\eta}{2}} \rightarrow 0 \quad \text{as } L \rightarrow \infty,$$

and consequently

$$C_{L+n}(\eta)\rho^{L+n+1} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus, applying (2.11), the proof of (2.9) is complete.

Now, let us focus on the Turán type inequality (2.10). If we choose $n = 0$ in (2.9), then we obtain

$${}_1\Delta_{L,\eta}(\rho) = \left(1 - \frac{C_{L+1,\eta}(\rho)}{C_{L,\eta}(\rho)}\right) F_L^2(\eta, \rho) - \sum_{i=1}^{\infty} \frac{B_{L+1,\eta}(\rho) \cdots B_{L+i+1,\eta}(\rho)}{C_{L,\eta}(\rho) \cdots C_{L+i,\eta}(\rho)} \Theta(B_{L+i-1,\eta}(\rho) C_{L+i,\eta}(\rho)) F_{L+i}^2(\eta, \rho).$$

In what follows we show that

$$(2.12) \quad \Theta(B_{L+i-1,\eta}(\rho) C_{L+i,\eta}(\rho)) = B_{L+i,\eta}(\rho) C_{L+i+1,\eta}(\rho) - B_{L+i-1,\eta}(\rho) C_{L+i,\eta}(\rho) \leq 0$$

for all $L \geq 0$, $\eta \leq 0$, $\rho > 0$ and $i \in \{1, 2, \dots\}$. Observe that the above inequality can be written as

$$\frac{(L+i)(L+i+2)((L+i+1)^2+\eta^2)}{(2L+2i+3)((L+i+1)(L+i+2)+\rho\eta)} \leq \frac{(L+i-1)(L+i+1)((L+i)^2+\eta^2)}{(2L+2i-1)((L+i-1)(L+i)+\rho\eta)},$$

which by using the notation $\omega = L+i$, can be rewritten as

$$\frac{\omega_1(\omega_2+\eta^2)}{\omega_3(\omega_4+\rho\eta)} \leq \frac{\omega_5(\omega_6+\eta^2)}{\omega_7(\omega_8+\rho\eta)}$$

where $\omega_1 = \omega(\omega+2)$, $\omega_2 = (\omega+1)^2$, $\omega_3 = 2\omega+3$, $\omega_4 = (\omega+1)(\omega+2)$, $\omega_5 = (\omega-1)(\omega+1)$, $\omega_6 = \omega^2$, $\omega_7 = 2\omega-1$ and $\omega_8 = (\omega-1)\omega$. Thus, in order to show (2.12) we need to verify the inequality

$$(\omega_3\omega_5 - \omega_1\omega_7)\rho\eta^3 + (\omega_3\omega_4\omega_5 - \omega_1\omega_7\omega_8)\eta^2 + (\omega_3\omega_5\omega_6 - \omega_1\omega_2\omega_7)\rho\eta + \omega_3\omega_4\omega_5\omega_6 - \omega_1\omega_2\omega_7\omega_8 \geq 0,$$

where $L \geq 0$, $\eta \leq 0$, $\rho > 0$ and $i \in \{1, 2, \dots\}$. Computations show that for all $L \geq 0$ and $i \in \{1, 2, \dots\}$ we have

$$\begin{cases} \omega_3\omega_5 - \omega_1\omega_7 = -3 < 0 \\ \omega_3\omega_4\omega_5 - \omega_1\omega_7\omega_8 = (\omega-1)(\omega+2)(8\omega^2+8\omega+3) \geq 0 \\ \omega_3\omega_5\omega_6 - \omega_1\omega_2\omega_7 = -2\omega(\omega+1)(2\omega^2+2\omega-1) < 0 \\ \omega_3\omega_4\omega_5\omega_6 - \omega_1\omega_2\omega_7\omega_8 = 4(\omega-1)\omega^2(\omega+1)^2(\omega+2) \geq 0 \end{cases},$$

which in turn implies the validity of inequality (2.12).

Now, by using the inequality (2.12) we obtain

$${}_1\Delta_{L,\eta}(\rho) \geq \left(1 - \frac{C_{L+1,\eta}(\rho)}{C_{L,\eta}(\rho)}\right) F_L^2(\eta, \rho),$$

where $L \geq 0$, $\eta \leq 0$ and $\rho > 0$. On the other hand for $\eta \leq 0$ and $L \geq 0$ the function

$$\rho \mapsto 1 - \frac{C_{L+1,\eta}(\rho)}{C_{L,\eta}(\rho)} = 1 - \frac{(L+2)(2L+1)\sqrt{(L+1)^2+\eta^2}}{(L+1)(2L+3)\sqrt{L^2+\eta^2}} \frac{L(L+1)+\rho\eta}{(L+1)(L+2)+\rho\eta}$$

is increasing on $(0, \infty)$ and consequently for all $L \geq 0$, $\eta \leq 0$ and $\rho > 0$ we have

$$1 - \frac{C_{L+1,\eta}(\rho)}{C_{L,\eta}(\rho)} \geq \lim_{\rho \rightarrow 0} \left[1 - \frac{C_{L+1,\eta}(\rho)}{C_{L,\eta}(\rho)}\right] = 1 - \frac{L(2L+1)\sqrt{(L+1)^2+\eta^2}}{(L+1)(2L+3)\sqrt{L^2+\eta^2}}$$

and this together with the above Turán type inequality gives (2.10).

Finally, let us consider the sharpness of (2.10). By using the relation [AS, p. 538]

$$L(2L+1)C_L(\eta) = \sqrt{L^2+\eta^2}C_{L-1}(\eta),$$

we obtain

$$\lim_{\rho \rightarrow 0} \frac{{}_1\Delta_{L,\eta}(\rho)}{F_L^2(\eta, \rho)} = 1 - \frac{C_{L-1}(\eta)C_{L+1}(\eta)}{C_L^2(\eta)} = 1 - \frac{L(2L+1)\sqrt{(L+1)^2 + \eta^2}}{(L+1)(2L+3)\sqrt{L^2 + \eta^2}}$$

and this shows that the above constant (depending on L and η) is best possible in (2.10). \square

2.2. Mitrinović-Adamović and Wilker type inequalities for Coulomb wave functions. Now, we present an immediate consequence of the Turán type inequality (2.10). For this consider the power series representation of the Coulomb wave function, namely [AS, p. 538]

$$F_L(\eta, \rho) = C_L(\eta) \sum_{n \geq 0} a_{L,n} \rho^{n+L+1},$$

where

$$a_{L,0} = 1, \quad a_{L,1} = \frac{\eta}{L+1} \quad \text{and} \quad a_{L,n} = \frac{2\eta a_{L,n-1} - a_{L,n-2}}{n(n+2L+1)}, \quad n \in \{2, 3, \dots\}.$$

Observe that the Turán type inequality (2.10) is equivalent to

$$(2.13) \quad \mathcal{F}_L^2(\eta, \rho) - \mathcal{F}_{L-1}(\eta, \rho)\mathcal{F}_{L+1}(\eta, \rho) \geq 0,$$

where $L, \rho > 0$, $\eta \leq 0$ and $\mathcal{F}_L(\eta, \rho)$ stands for the normalized regular Coulomb wave function, defined by

$$\mathcal{F}_L(\eta, \rho) = C_L^{-1}(\eta) \rho^{-L-1} F_L(\eta, \rho) = \sum_{n \geq 0} a_{L,n} \rho^n.$$

Theorem 3. *If $\eta \leq 0$, $L > -1$ and $0 < \rho < x_{L,\eta,1}$, then the following Mitrinović-Adamović and Wilker type inequalities are valid*

$$(2.14) \quad [\mathcal{F}_L(\eta, \rho)]^{L+\frac{3}{2}} < [\mathcal{F}_{L+1}(\eta, \rho)]^{L+\frac{5}{2}},$$

$$(2.15) \quad [\mathcal{F}_{L+1}(\eta, \rho)]^{\frac{1}{L+\frac{3}{2}}} + \frac{\mathcal{F}_{L+1}(\eta, \rho)}{\mathcal{F}_L(\eta, \rho)} \geq 2.$$

We note that if we choose $\eta = 0$ and $L+1/2 = \nu$ in Theorem 3, then we reobtain the next Mitrinović-Adamović and Wilker type inequalities [Ba, Theorem 3]

$$\mathcal{J}_\nu^{\nu+1}(\rho) \leq \mathcal{J}_{\nu+1}^{\nu+2}(\rho) \quad \text{and} \quad [\mathcal{J}_{\nu+1}(\rho)]^{\frac{1}{\nu+1}} + \frac{\mathcal{J}_{\nu+1}(\rho)}{\mathcal{J}_\nu(\rho)} \geq 2,$$

where $\nu > -1/2$ and $0 < \rho < j_{\nu,1}$. Here $x_{L,0,n} = j_{\nu,n}$ stands for the n th positive zero of the Bessel function J_ν , and \mathcal{J}_ν stands for the normalized Bessel function, defined by $\mathcal{F}_L(0, \rho) = \mathcal{J}_\nu(\rho) = 2^\nu \Gamma(\nu+1) \rho^{-\nu} J_\nu(\rho)$. It is important to note here that the above inequalities are valid for all $\nu > -1$ and the case $\nu = -1/2$ corresponds to the original Mitrinović-Adamović and Wilker inequalities for sine and cosine functions. See [Ba, BS, WB] for more details on Mitrinović-Adamović and Wilker inequalities.

Proof of Theorem 3. Consider the function $\varphi_L(\eta, \rho)$, defined by

$$\varphi_L(\eta, \rho) = \left(L + \frac{5}{2}\right) \log [\mathcal{F}_{L+1}(\eta, \rho)] - \left(L + \frac{3}{2}\right) \log [\mathcal{F}_L(\eta, \rho)].$$

Observe that the above function is well defined since for each $\eta \leq 0$, $L > -1$ and $0 < \rho < x_{L,\eta,1}$ we have

$$\mathcal{F}_L(\eta, \rho) > 0 \quad \text{and} \quad \mathcal{F}_{L+1}(\eta, \rho) > 0.$$

Now, by using the recurrence relation (2.3) we obtain

$$(2.16) \quad \mathcal{F}'_L(\eta, \rho) = \frac{\eta}{L+1} \mathcal{F}_L(\eta, \rho) - \frac{(L+1)^2 + \eta^2}{(L+1)^2(2L+3)} \rho \mathcal{F}_{L+1}(\eta, \rho),$$

and consequently

$$2\varphi'_L(\eta, \rho) = \eta \left(\frac{2L+5}{L+2} - \frac{2L+3}{L+1} \right) + \frac{1}{\mathcal{F}_{L+1}^2(\eta, \rho)} \frac{\rho \mathcal{F}_{L+1}(\eta, \rho)}{\mathcal{F}_L(\eta, \rho)} \Phi_L(\eta, \rho),$$

where according to (2.13)

$$\begin{aligned} \Phi_L(\eta, \rho) &= \frac{(L+1)^2 + \eta^2}{(L+1)^2} \mathcal{F}_{L+1}^2(\eta, \rho) - \frac{(L+2)^2 + \eta^2}{(L+2)^2} \mathcal{F}_L(\eta, \rho) \mathcal{F}_{L+2}(\eta, \rho) \\ &\geq \frac{(L+2)^2 + \eta^2}{(L+2)^2} [\mathcal{F}_{L+1}^2(\eta, \rho) - \mathcal{F}_L(\eta, \rho) \mathcal{F}_{L+2}(\eta, \rho)] \geq 0. \end{aligned}$$

On the other hand, by using the Mittag-Leffler expansion (2.1) we obtain

$$\frac{\rho \mathcal{F}_{L+1}(\eta, \rho)}{\mathcal{F}_L(\eta, \rho)} = \frac{(L+1)^2(2L+3)}{(L+1)^2 + \eta^2} \sum_{n \geq 1} \left[\frac{\rho}{x_{L,\eta,n}(x_{L,\eta,n} - \rho)} + \frac{\rho}{y_{L,\eta,n}(y_{L,\eta,n} - \rho)} \right] > 0,$$

where $\eta \leq 0$, $L > -1$ and $0 < \rho < x_{L,\eta,1}$. These imply that for those values of η, ρ, L we have $\varphi'_L(\eta, \rho) \geq 0$ and thus

$$\varphi_L(\eta, \rho) \geq \varphi_L(\eta, 0) = 0,$$

which completes the proof of (2.14). Finally, the Wilker type inequality (2.15) follows immediately from the inequality (2.14) and the arithmetic-geometric mean inequality for the values $[\mathcal{F}_{L+1}(\eta, \rho)]^{1/(L+3/2)}$ and $\mathcal{F}_{L+1}(\eta, \rho)/\mathcal{F}_L(\eta, \rho)$, that is,

$$[\mathcal{F}_{L+1}(\eta, \rho)]^{\frac{1}{L+\frac{3}{2}}} + \frac{\mathcal{F}_{L+1}(\eta, \rho)}{\mathcal{F}_L(\eta, \rho)} \geq 2 \sqrt{[\mathcal{F}_{L+1}(\eta, \rho)]^{\frac{1}{L+\frac{3}{2}}} \cdot \frac{\mathcal{F}_{L+1}(\eta, \rho)}{\mathcal{F}_L(\eta, \rho)}} \geq 2.$$

□

2.3. Some properties of Coulomb zeta functions. This subsection is devoted to the study of some functions involving the positive and negative zeros of Coulomb wave functions. We give some basic properties, like recurrence relations, monotonicity properties and we study the higher order derivatives of these functions. We note that some of the results were already obtained in [ŠŠ], but here we use a different approach.

For $s > 1$ and $L, \eta \in \mathbb{R}$ let us consider functions $X_{s,\eta}(L)$, $Y_{s,\eta}(L)$ and $\zeta_{s,\eta}(L)$, which we call as the Coulomb zeta functions, defined by

$$X_{s,\eta}(L) = \sum_{n \geq 1} \frac{1}{x_{L,\eta,n}^s}, \quad Y_{s,\eta}(L) = \sum_{n \geq 1} \frac{1}{y_{L,\eta,n}^s} \quad \text{and} \quad \zeta_{s,\eta}(L) = X_{s,\eta}(L) + Y_{s,\eta}(L).$$

By using the Mittag-Leffler expansion (2.1) we obtain for all $0 < \rho < \min\{x_{L,\eta,1}, -y_{L,\eta,1}\}$ the generating function for the Coulomb zeta functions as follows

$$\begin{aligned} \frac{\rho F_{L+1}(\eta, \rho)}{F_L(\eta, \rho)} &= \frac{L+1}{\sqrt{(L+1)^2 + \eta^2}} \sum_{n \geq 1} \left[\frac{\left(\frac{\rho}{x_{L,\eta,n}}\right)^2}{1 - \frac{\rho}{x_{L,\eta,n}}} + \frac{\left(\frac{\rho}{y_{L,\eta,n}}\right)^2}{1 - \frac{\rho}{y_{L,\eta,n}}} \right] \\ &= \frac{L+1}{\sqrt{(L+1)^2 + \eta^2}} \sum_{n \geq 1} \left[\sum_{m \geq 0} \left(\frac{\rho}{x_{L,\eta,n}}\right)^{m+2} + \sum_{m \geq 0} \left(\frac{\rho}{y_{L,\eta,n}}\right)^{m+2} \right] \\ &= \frac{L+1}{\sqrt{(L+1)^2 + \eta^2}} \sum_{m \geq 0} [X_{m+2,\eta}(L) + Y_{m+2,\eta}(L)] \rho^{m+2}, \end{aligned}$$

that is, we have

$$(2.17) \quad \frac{F_{L+1}(\eta, \rho)}{\rho F_L(\eta, \rho)} = \frac{L+1}{\sqrt{(L+1)^2 + \eta^2}} \sum_{m \geq 0} \zeta_{m+2,\eta}(L) \rho^m.$$

Let us suppose that $\eta = 0$. Then $x_{L,0,n} = -y_{L,0,n} = j_{L+1/2,n}$ for all $n \in \{1, 2, \dots\}$ and the formula (2.17) reduces to

$$\frac{\rho J_{L+3/2}(\rho)}{J_{L+1/2}(\rho)} = \sum_{m \geq 0} \zeta_{m+2,0}(L) \rho^{m+2} = 2 \sum_{k \geq 1} \left[\sum_{n \geq 1} \frac{1}{j_{L+1/2,n}^{2k}} \right] \rho^{2k}.$$

Now, let $L + 1/2$ be denoted by ν , then for $|\rho| < j_{\nu,1}$ we obtain the Kishore's formula [Ki, p. 528]

$$\frac{\rho J_{\nu+1}(\rho)}{2J_{\nu}(\rho)} = \sum_{k \geq 1} \sigma_{2k}(\nu) \rho^{2k},$$

where

$$\sigma_{2k}(\nu) = X_{2k,0}(\nu - 1/2) = \sum_{n \geq 1} \frac{1}{j_{\nu,n}^{2k}}$$

is the so-called Rayleigh function. Observe that

$$\lim_{\rho \rightarrow 0} \left[\frac{F_{L+1}(\eta, \rho)}{\rho F_L(\eta, \rho)} \right] = \frac{C_{L+1}(\eta)}{C_L(\eta)} = \frac{\sqrt{(L+1)^2 + \eta^2}}{(L+1)(2L+3)},$$

and consequently if $\rho \rightarrow 0$ in (2.17), then we obtain

$$\zeta_{2,\eta}(L) = \frac{(L+1)^2 + \eta^2}{(L+1)^2(2L+3)}.$$

It is also worth to mention that if we use (2.17) and the power series representation of the Coulomb wave function, then we obtain

$$\begin{aligned} C_{L+1}(\eta) (a_{L+1,0} + a_{L+1,1}\rho + \dots + a_{L+1,n}\rho^n + \dots) &= \frac{L+1}{\sqrt{(L+1)^2 + \eta^2}} C_L(\eta) \\ &\times (a_{L,0} + a_{L,1}\rho + \dots + a_{L,n}\rho^n + \dots) (\zeta_{2,\eta}(L) + \zeta_{3,\eta}(L)\rho + \dots + \zeta_{n+2,\eta}(L)\rho^n + \dots), \end{aligned}$$

and identifying the coefficients of ρ^n on both sides we arrive at the recurrence relation

$$\zeta_{2,\eta}(L) a_{L+1,n} = \sum_{k=0}^n a_{L,k} \zeta_{n-k+2,\eta}(L), \quad n \in \{0, 1, \dots\}.$$

By using the above relation for $n = 1$ we obtain

$$\zeta_{3,\eta}(L) = -\eta \frac{(L+1)^2 + \eta^2}{(L+1)^3(L+2)(2L+3)},$$

and other values of $\zeta_{m,\eta}(L)$ can be computed also for $m \in \{4, 5, \dots\}$. Moreover, by using the relations (2.3) and (2.17) we obtain

$$\frac{F'_L(\eta, \rho)}{F_L(\eta, \rho)} = \frac{L+1}{\rho} + \frac{\eta}{L+1} - \sum_{m \geq 0} \zeta_{m+2,\eta}(L) \rho^{m+1}$$

and taking this in (2.7) and identifying the coefficients of ρ^m on both sides we obtain

$$(2.18) \quad (m+2L+3)\zeta_{m+2,\eta}(L) + \frac{2\eta}{L+1}\zeta_{m+1,\eta}(L) = \sum_{k=2}^m \zeta_{k,\eta}(L)\zeta_{m-k+2,\eta}(L), \quad m \in \{2, 3, \dots\}.$$

Observe that the above result implies that the Coulomb zeta functions are actually rational functions of L . We mention that the above results were obtained also by Štampach and Šťovíček [ŠŠ], however, they used a different approach.

Now, we are ready to prove the following new result by using (2.18).

Theorem 4. *If $\eta \leq 0$ and $m \in \{2, 3, \dots\}$, then the Coulomb zeta function $L \mapsto \zeta_{m,\eta}(L)$, as well as the functions $L \mapsto (m+2L+3)\zeta_{m+2,\eta}(L) + 2\eta\zeta_{m+1,\eta}(L)/(L+1)$, $L \mapsto \zeta_{m,\eta}(L)/\zeta_{2,\eta}(L)$ and $L \mapsto (2L+3)^{m-1}\zeta_{m,\eta}(L)$ are completely monotonic on $(-1, \infty)$.*

Let $\eta = 0$. Then $x_{L,0,n} = -y_{L,0,n} = j_{L+1/2,n}$ for all $n \in \{1, 2, \dots\}$ and

$$\zeta_{s,0}(L) = \sum_{n \geq 1} \frac{(-1)^s + 1}{(-1)^s} \frac{1}{j_{L+1/2,n}^s}.$$

Observe that for all $s > 1$ we have $\zeta_{2s,0}(L) = 2X_{2s,0}(L)$ and $\zeta_{2s-1,0}(L) = 0$. Now, taking $m = 2r$ in (2.18) we obtain

$$(2r+2L+3)\zeta_{2r+2,0}(L) = \sum_{k=1}^r \zeta_{2k,0}(L)\zeta_{2r-2k+2,0}(L),$$

and if we let $L+1/2 = \nu$ and $r+1 = q$, then the above relation becomes

$$(\nu+q)\sigma_{2q}(\nu) = \sum_{k=1}^{q-1} \sigma_{2k}(\nu)\sigma_{2q-2k}(\nu),$$

which is the result of Kishore [Ki, p. 532]. We also note here that in particular when $\eta = 0$ the results of Theorem 4 reduce to the main results of Obi [Ob, p. 466] concerning the complete monotonicity of the functions $\nu \mapsto \sigma_{2q}(\nu)$, $\nu \mapsto (\nu+1)^q \sigma_{2q}(\nu)$ and $(\nu+q)\sigma_{2q}(\nu)$ on $(-1/2, \infty)$, where $q \in \{1, 2, \dots\}$.

Proof of Theorem 4. Since the sum and product of completely monotonic functions are also completely monotonic, we have that for $\eta \leq 0$ the functions $L \mapsto \zeta_{2,\eta}(L)$ and $L \mapsto \zeta_{3,\eta}(L)$ are completely monotonic on $(-1, \infty)$. On the other hand, from (2.18) we have

$$\zeta_{m+2,\eta}(L) = -\frac{2\eta}{(L+1)(m+2L+3)}\zeta_{m+1,\eta}(L) + \frac{1}{m+2L+3} \sum_{k=2}^m \zeta_{k,\eta}(L)\zeta_{m-k+2,\eta}(L), \quad m \in \{2, 3, \dots\}.$$

Thus, if we suppose that $L \mapsto \zeta_{s,\eta}(L)$ is completely monotonic on $(-1, \infty)$ for each $s \in \{2, 3, \dots, m+1\}$, then by induction we get that $L \mapsto \zeta_{m+2,\eta}(L)$ is also completely monotonic on $(-1, \infty)$.

Similarly, the functions $L \mapsto (2L+3)\zeta_{2,\eta}(L)$ and $L \mapsto (2L+3)^2\zeta_{3,\eta}(L)$ are clearly completely monotonic on $(-1, \infty)$ for all $\eta \leq 0$. Supposing that $L \mapsto (2L+3)^{s-1}\zeta_{s,\eta}(L)$ is completely monotonic on $(-1, \infty)$ for each $s \in \{2, 3, \dots, m+1\}$, the relation

$$\begin{aligned} (2L+3)^{m+1}\zeta_{m+2,\eta}(L) &= -\frac{2\eta(2L+3)^m}{(L+1)(m+2L+3)}\zeta_{m+1,\eta}(L) \\ &\quad + \frac{1}{m+2L+3} \sum_{k=2}^m [(2L+3)^{k-1}\zeta_{k,\eta}(L)] [(2L+3)^{m-k+1}\zeta_{m-k+2,\eta}(L)], \quad m \in \{2, 3, \dots\}. \end{aligned}$$

and complete mathematical induction imply that $L \mapsto (2L+3)^{m+1}\zeta_{m+2,\eta}(L)$ is also completely monotonic on $(-1, \infty)$.

Observe that for $\eta \leq 0$ the functions

$$\begin{aligned} L \mapsto \frac{\zeta_{3,\eta}(L)}{\zeta_{2,\eta}(L)} &= -\frac{\eta}{(L+1)(L+2)}, \\ L \mapsto \frac{\zeta_{4,\eta}(L)}{\zeta_{2,\eta}(L)} &= \frac{(L+2)(L+1)^2 + (5L+8)\eta^2}{(L+1)^2(L+2)(2L+3)(2L+5)} \end{aligned}$$

are completely monotonic on $(-1, \infty)$. If $L \mapsto \zeta_{s,\eta}(L)/\zeta_{2,\eta}(L)$ is completely monotonic on $(-1, \infty)$ for $s \in \{2, 3, \dots, m+1\}$, then in view of

$$\frac{\zeta_{m+2,\eta}(L)}{\zeta_{2,\eta}(L)} = -\frac{2\eta}{(L+1)(m+2L+3)} \frac{\zeta_{m+1,\eta}(L)}{\zeta_{2,\eta}(L)} + \frac{1}{m+2L+3} \sum_{k=2}^m \frac{\zeta_{k,\eta}(L)}{\zeta_{2,\eta}(L)} \zeta_{m-k+2,\eta}(L), \quad m \in \{2, 3, \dots\}$$

and by using the fact that $L \mapsto \zeta_{s,\eta}(L)$ is completely monotonic on $(-1, \infty)$ for all $s \in \{2, 3, \dots, m\}$, by using mathematical induction we obtain that $L \mapsto \zeta_{m+2,\eta}(L)/\zeta_{2,\eta}(L)$ is also completely monotonic on $(-1, \infty)$.

Finally, the first part of this theorem together with (2.18) imply that the function

$$L \mapsto (m+2L+3)\zeta_{m+2,\eta}(L) + 2\eta\zeta_{m+1,\eta}(L)/(L+1)$$

is also completely monotonic on $(-1, \infty)$ for all $m \in \{2, 3, \dots\}$ and $\eta \leq 0$. \square

2.4. Interlacing properties of the zeros of Coulomb wave functions. The first part of the next result is the extension of a result of Miyazaki et al. [MKCI, Remark 4.3], which states that if $\rho > 0$, $\eta \in \mathbb{R}$ and $L \in \{1, 2, \dots\}$, then there is one and only one zero of $\rho \mapsto F'_L(\eta, \rho)$ between two continuous zeros of $\rho \mapsto F_L(\eta, \rho)$.

Theorem 5. *If $L > -1/2$ and $\eta \in \mathbb{R}$, then the zeros of $\rho \mapsto F_L(\eta, \rho)$ and $\rho \mapsto F'_L(\eta, \rho)$ are interlacing. Moreover, if $L > -1$ and $\eta \in \mathbb{R}$, then the zeros of $\rho \mapsto F_L(\eta, \rho)$ and $\rho \mapsto \rho F'_L(\eta, \rho) - (L+1)F_L(\eta, \rho)$ are interlacing.*

Proof. In view of (2.8), for $L > -1$ the function $\rho \mapsto F'_L(\eta, \rho)/F_L(\eta, \rho)$ is decreasing on the interval $(x_{L,\eta,k}, x_{L,\eta,k+1})$, where $k \in \{1, 2, \dots\}$. Moreover, the expression $F'_L(\eta, \rho)/F_L(\eta, \rho)$ tends to $-\infty$ as $\rho \nearrow x_{L,\eta,k+1}$ and tends to ∞ as $\rho \searrow x_{L,\eta,k}$. Since [ŠŠ, Remark 17] for $L > -1/2$ and $\eta \in \mathbb{R}$ the zeros of $\rho \mapsto F'_L(\eta, \rho)$ are real and simple, it follows that $\rho \mapsto F'_L(\eta, \rho)/F_L(\eta, \rho)$ intersects once and only once the horizontal axis, and the abscissa of the intersection point is actually the k th positive zero of $\rho \mapsto F'_L(\eta, \rho)$. The interlacing property of the negative zeros is similar, and thus we omit the details.

Hadamard's theorem states that an entire function of finite order τ may be represented in the form

$$f(z) = z^m e^{P_q(z)} \prod_{n \geq 1} G\left(\frac{z}{a_n}, p\right),$$

where a_1, a_2, \dots are all nonzero roots of $f(z)$, $p \leq \tau$, $P_q(z)$ is a polynomial in z of degree $q \leq \tau$, m is the multiplicity of the root at the origin, and $G(u, p) = (1 - u)e^{u + \frac{u^2}{2} + \dots + \frac{u^p}{p}}$ for $p > 0$. Combining this with (2.2) it follows that the growth order τ_C of the normalized entire Coulomb wave function $\rho \mapsto \mathcal{F}_L(\eta, \rho)$ satisfies $1 \leq \tau_C < 2$. It is known that the genus of an entire function of order τ is $[\tau]$ when τ is not an integer, but the genus of an entire function of natural order τ can be either τ or $\tau - 1$. Thus, the normalized entire Coulomb wave function $\rho \mapsto \mathcal{F}_L(\eta, \rho)$ is of genus 0 or 1. On the other hand, Laguerre's theorem on separation of zeros states that, if $z \mapsto f(z)$ is an entire function, not a constant, which is real for real z and has only real zeros, and is of genus 0 or 1, then the zeros of f' are also real and are separated by the zeros of f . According to [ŠŠ, Proposition 13] when $L > -1$ and $\eta \in \mathbb{R}$ the zeros of $\rho \mapsto \mathcal{F}_L(\eta, \rho)$ are all real. Thus, appealing on Laguerre's separation theorem we conclude that when $L > -1$ and $\eta \in \mathbb{R}$, then the zeros of $\rho \mapsto \rho F'_L(\eta, \rho) - (L + 1)F_L(\eta, \rho)$ are all real and are interlacing with the zeros of $\rho \mapsto F_L(\eta, \rho)$. \square

REFERENCES

- [AS] M. ABRAMOWITZ, I.A. STEGUN (Eds.), *Handbook of Mathematical Functions with formulas. Graphs and Mathematical Tables*, Dover Publications, New York, 1965.
- [Ba] Á. BARICZ, Functional inequalities involving Bessel and modified Bessel functions of the first kind, *Expo. Math.* 26 (2008) 279–293.
- [BP] Á. BARICZ, T.K. POGÁNY, Turán determinants of Bessel functions, *Forum Math.* 26 (2014) 295–322.
- [BS] Á. BARICZ, J. SÁNDOR, Extensions of the generalized Wilker inequality to Bessel functions, *J. Math. Inequal.* 2(3) (2008) 397–406.
- [BI] J. BUSTOZ, M.E.H. ISMAIL, Turán inequalities for symmetric orthogonal polynomials, *Internat. J. Math. Math. Sci.* 20 (1997) 1–8.
- [Ik] Y. IKEBE, The zeros of regular Coulomb wave functions and of their derivatives, *Math. Comp.* 29(131) (1975) 878–887.
- [JB] C.M. JOSHI, S.K. BISSU, Some inequalities of Bessel and modified Bessel functions, *J. Austral. Math. Soc. (Series A)* 50 (1991) 333–342.
- [KS] S. KARLIN, G. SZEGŐ, On certain determinants whose elements are orthogonal polynomials, *J. Analyse Math.* 8 (1960/61), 1–157.
- [Ki] N. KISHORE, The Rayleigh function, *Proc. Amer. Math. Soc.* 14 (1963) 527–533.
- [La] S.K. LAKSHMANA RAO, On the relative extrema of the Turán expression for Bessel functions, *Proc. Indian Acad. Sci. Sect. A* 53 (1961) 239–243.
- [MKCI] Y. MIYAZAKI, Y. KIKUCHI, D. CAI, Y. IKEBE, Error analysis for the computation of zeros of regular Coulomb wave function and its first derivative, *Math. Comp.* 70(235) (2001) 1195–1204.
- [Ni] T. NISHIYAMA, Application of Coulomb wave functions to an orthogonal series associated with steady axisymmetric Euler flows, *J. Approx. Theory* 151 (2008) 42–59.
- [Ob] E.C. OBI, The complete monotonicity of the Rayleigh function, *J. Math. Anal. Appl.* 77 (1980) 465–468.
- [Pa] M.L. PATRICK, Extensions of inequalities of the Laguerre and Turán type, *Pacific J. Math.* 44 (1973) 675–682.
- [Ro] D.K. ROSS, Inequalities and identities for $y_n^2 - y_{n-1}y_{n+1}$, *Aequat. Math.* 20 (1980) 23–32.
- [Sk] H. SKOVGAARD, On inequalities of the Turán type, *Math. Scand.* 2 (1954) 65–73.
- [ŠŠ] F. ŠTAMPACH, P. ŠTOVÍČEK, Orthogonal polynomials associated with Coulomb wave functions, *J. Math. Anal. Appl.* 419(1) (2014) 231–254.
- [Sz1] O. SZÁSZ, Inequalities concerning ultraspherical polynomials and Bessel functions, *Proc. Amer. Math. Soc.* 1 (1950) 256–267.
- [Sz2] O. SZÁSZ, Identities and inequalities concerning orthogonal polynomials and Bessel functions, *J. Analyse Math.* 1 (1951) 116–134.
- [TN] V.R. THIRUVENKATACHAR, T.S. NANJUNDIAH, Inequalities concerning Bessel functions and orthogonal polynomials, *Proc. Indian Acad. Sci. Sect. A* 33 (1951) 373–384.
- [Wi] J. WIMP, Some explicit Padé approximants for the function Φ'/Φ and a related quadrature formula involving Bessel functions, *SIAM J. Math. Anal.* 16(4) (1985) 887–895.
- [WB] S. WU, Á. BARICZ, Generalizations of Mitrinović, Adamović and Lazarević's inequalities and their applications, *Publ. Math. Debrecen* 75 (2009) 447–458.

INSTITUTE OF APPLIED MATHEMATICS, ÓBUDA UNIVERSITY, BUDAPEST, HUNGARY

DEPARTMENT OF ECONOMICS, BABEȘ-BOLYAI UNIVERSITY, CLUJ-NAPOCA, ROMANIA

E-mail address: bariczocsi@yahoo.com